

**ON THE CALCULATION OF  
DERIVATIVES OF EIGENVALUES AND EIGENVECTORS  
IN THE SIMULTANEOUS DESIGN AND CONTROL OF STRUCTURES**

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## ABSTRACT

Independent Modal Space Control (IMSC) is a technique that is often used for the control of large order structural systems. The pertinent optimization problem in the simultaneous design and control of structures is a min - min problem that minimizes with respect to the structural design variables, the minimum value of the performance index with respect to the control forces obtained using the IMSC technique. The minimization process requires derivatives of eigenvalues and eigenvectors with respect to the design variables. These derivatives can be computed by a rather involved analytical procedure or a relatively simple finite difference procedure. This paper examines the computer cost effectiveness of these two procedures for the derivative calculations.

## INTRODUCTION

One of the objectives of structural control is to suppress undesirable motion resulting from some unavoidable excitation such as onboard machinery or docking maneuvers. In active control the motion of structure is sensed and suitable forces are applied to reduce and ultimately eliminate the undesirable motion. In optimal control the forces are applied such that a preselected performance index is minimized. The solution of the optimal control problem requires the solution of the matrix Ricatti equation. Because of the difficulties encountered in numerical computations, the solution of the matrix Ricatti equation is not feasible for large order systems. For large order systems, an alternate method known as the Independent Modal Space Control (IMSC) [1] is more suitable.

In the IMSC method, the control forces are specified in the modal space instead of in the physical space. Also by suitably choosing the modal control forces, each mode of vibration is controlled independently of the other modes. The performance index is assumed to be of the form

$$J = \sum_{r=1}^{\ell} J_r \quad (1)$$

where  $\ell$  is the number of modes controlled and  $J_r$  is the performance index associated with the  $r$ -th mode and has the definition

$$J_r = \int_0^{t_f} ( \omega_r^2 \xi_r^2 + \omega_r^2 \eta_r^2 + R_r z_{cr}^2 ) dt \quad (2)$$

where

$$\eta_r = \dot{\xi}_r / \omega_r \quad , \quad (3)$$

$\omega_r$  is the frequency of the  $r$ th mode and  $R_r > 0$  is the penalty parameter imposed on the control effort. A higher value of  $R_r$  will result in a smaller control force in the modal space and vice versa.

The modal coordinates  $\xi_r$ ,  $r = 1, \dots, \ell$  are related to the displacement vector  $u$ , by the relation

$$u = X\xi \quad (4)$$

where  $X$  is the modal matrix, having as its columns the eigenvectors, obtained by the solution of the eigenvalue problem

$$K X = \omega^2 M X \quad (5)$$

$\xi_r$  and  $\eta_r$  satisfy the constraint equations

$$\ddot{\xi}_r(t) + \omega_r^2 \xi_r(t) = z_{cr}(t) \quad (6.a)$$

$$\xi_r(0) = \xi_{r0} \quad (6.b)$$

$$\dot{\xi}_r(0) = \eta_r(0) \omega_r \quad (6.c)$$

where  $Z_c = X^T F \quad (7)$

is the modal control vector. Minimization of  $J_r$  in Eq. (2) with the differential constraint equations given by Eqs. (6) leads to a  $2 \times 2$  matrix Riccati equation that can be solved analytically for  $t_f = \infty$ .

For this case, the control force is given by [2]

$$\begin{aligned} z_{cr}(t) = & \omega_r \left( \omega_r - \left( \omega_r^2 + R_r^{-1} \right)^{1/2} \right) \xi_r(t) \\ & - \left[ 2 \omega_r \left( -\omega_r + \left( \omega_r^2 + R_r^{-1} \right)^{1/2} \right) + R_r^{-1} \right]^{1/2} \dot{\xi}_r(t) \end{aligned} \quad (8)$$

and the solution of the closed loop modal equations (6) for the controlled modes gives

$$\xi_r(t) = e^{-a_1 t} (\alpha_1 \cos \theta t + \beta_1 \sin \theta t) \quad (9)$$

with

$$a_1 = \lambda_r \omega_r \quad (10)$$

$$\theta = (\omega_d)_r \quad (11)$$

$$\alpha_1 = \xi_{r0} \quad (12)$$

$$\beta_1 = \frac{\omega_r}{(\omega_d)_r} (\xi_{r0} \lambda_r + \eta_r(0)) \quad (13)$$

$$\lambda_r = - \frac{f_{22}}{2\omega_r} \quad (14)$$

$$(\omega_d)_r = (\omega_r^2 - \omega_r f_{21} - \frac{f_{22}^2}{4})^{1/2} \quad (15)$$

$$f_{21} = - R_r^{-1} k_{21}, \quad f_{22} = - R_r^{-1} k_{22} \quad (16)$$

$$k_{21} = k_{12} = - \omega_r R_r + (\omega_r^2 R_r^2 + R_r)^{1/2} \quad (17)$$

$$k_{22} = (R_r - 2\omega_r^2 R_r^2 + 2\omega_r R_r (\omega_r^2 R_r^2 + R_r)^{1/2})^{1/2} \quad (18)$$

Substitution of Eqs. (9) through (18) into Eq. (2) followed by its integration with  $t_f = \infty$  yields

$$J_r = \frac{\omega_r^2}{2} (E I_{11} + F I_{22} + G I_{22}) \quad (19)$$

with

$$J_{11} = 1 + \frac{k_{12}^2}{\omega_r^2 R_r}, \quad J_{12} = \frac{2 k_{12} k_{22}}{\omega_r^2 R_r}, \quad J_{22} = 1 + \frac{k_{22}^2}{\omega_r^2 R_r} \quad (20)$$

$$K_{11} = \left[ \frac{1}{\omega_r^2} + \frac{2}{\omega_r^2 R_r} (\omega_r^2 R_r^2 + R_r)^{3/2} - 2 R_r^2 \omega_r^2 - R_r \right]^{1/2} \quad (21)$$

$$\alpha_2 = \eta_r(0) \quad (22)$$

$$\beta_2 = \frac{1}{(\omega_d)_r} [\xi_{ro} (f_{21} - \omega_r) - \eta_r(0) \omega_r \lambda_r] \quad (23)$$

$$E = J_{11} \alpha_1^2 + J_{22} \alpha_2^2 + J_{12} \alpha_1 \alpha_2 \quad (24)$$

$$F = J_{11} \beta_1^2 + J_{22} \beta_2^2 + J_{12} \beta_1 \beta_2 \quad (25)$$

$$G = 2 J_{11} \alpha_1 \beta_1 + 2 J_{22} \alpha_2 \beta_2 + J_{12} (\alpha_1 \beta_2 + \alpha_2 \beta_1) \quad (26)$$

$$I_{11} = \frac{1}{4 a_1} + \frac{a_1}{4 a_1^2 + \theta^2} \quad (27)$$

$$I_{22} = - \frac{a_1}{4 a_1^2 + \theta^2} + \frac{1}{4 a_1} \quad (28)$$

$$I_{12} = \frac{1}{4} \left( \frac{\theta}{a_1^2 + \theta^2} \right) \quad (29)$$

## Simultaneous Design and Control

The process of simultaneous design and control of structures is a min - min problem that minimizes with respect to the structural design variables the minimum value of the performance index with respect to the control forces.

The minimum value of the performance index with respect to control forces is given by Eq. (1) wherein  $J_r$  can be evaluated from Eqs. (8) through (29). In the process of minimization of this minimum value of the performance index, its derivatives with respect to the design variables are required. These derivatives can be evaluated explicitly by a laborious, even though straightforward differentiation of Eqs. (8) through (29) with respect to the design variables provided the derivatives of the eigenvalues  $\omega_r$  and the eigenvectors  $x_r$ ,  $r = 1, 2, \dots, \ell$  are available. The other alternative is to calculate the derivatives of  $J$  with respect to the design variables by using say the forward difference scheme. The latter is easily programmable since no explicit derivatives of the eigenvalues and eigenvectors with respect to the design variables are then required. The thrust of this paper is a comparison of the computational cost and the efficiency of the two procedures for calculating the derivatives of the performance index.

Before we elaborate on this comparison however, we will digress and discuss the calculation of the derivatives of the eigenvalues and eigenvectors using the well known Nelson's method [3].

### Derivatives of Eigenvalues and Eigenvectors with Respect to the Design Variables

Purely from a computer programming point of view the simplest and the most straightforward though not necessarily the most efficient way to compute the derivatives of eigenvalues and eigenvectors is by using finite differences in particular the forward difference scheme with an appropriate step size [4]. The main disadvantage of the forward difference scheme is that it requires the solution of an eigenvalue problem once for each design variable. This could be a computationally expensive process. Furthermore, to obtain an accurate value of the computed derivatives, the eigenvalue problems need to be solved with a high degree of precision.

The eigenvalue  $\omega_r^2$ , and the eigenvector  $x_r$ , of the previous section are obtained by the solution of the eigenvalue problem.

$$\omega_r^2 M x_r = K x_r, \quad r = 1, 2, \dots, \ell \quad (30)$$

where  $M$  and  $K$  are the assembled mass and stiffness matrices respectively of the finite-element model of the structure. The mode shape  $x_r$  is normalized with respect to the mass matrix  $M$  as

$$x_r^T M x_s = \delta_{rs} \quad (31)$$

wherein  $\delta_{rs}$  is the Kronecker delta. Differentiating Eqs. (30) and (31) with respect to a design variable  $p_j$  for a particular eigenpair  $(\omega_r^2, x_r)$  with distinct eigenvalues one obtains

$$(K - \omega_r^2 M) \left( \frac{dx_r}{dp_j} \right) - \left( \frac{d\omega_r^2}{dp_j} \right) M x_r = - \left( \frac{dK}{dp_j} - \omega_r^2 \frac{dM}{dp_j} \right) x_r \quad (32)$$

$$x_r^T M \frac{dx_r}{dp_j} = - \frac{1}{2} x_r^T \left( \frac{dM}{dp_j} \right) x_r \quad (33)$$

where use has been made of the symmetry of the mass matrix  $M$ .

To obtain the derivatives of eigenvalues, Eq. (32) is premultiplied by  $x_r^T$  followed by the use of Eq. (31) to yield

$$\frac{d\omega_r^2}{dp_j} = x_r^T \left( \frac{dK}{dp_j} - \omega_r^2 \frac{dM}{dp_j} \right) x_r \quad (34)$$

To obtain the derivatives of the eigenvector  $x_r$ , Eqs. (32) and (33) are combined as

$$\begin{bmatrix} K - \omega_r^2 M & - M x_r \\ - x_r^T M & 0 \end{bmatrix} \begin{Bmatrix} dx_r/dp_j \\ d\omega_r^2/dp_j \end{Bmatrix} = \begin{Bmatrix} - \left( \frac{dK}{dp_j} - \omega_r^2 \frac{dM}{dp_j} \right) x_r \\ - \frac{1}{2} x_r^T \left( \frac{dM}{dp_j} \right) x_r \end{Bmatrix} \quad (35)$$

Equations (35) could be solved for both the eigenvalue and the eigenvector derivatives except that the principal minor  $K - \omega_r^2 M$  is singular. To circumvent this apparent difficulty, Nelson [5] proposed a method that temporarily imposes the normalization equation (31) by the requirement that the largest component of the eigenvector be equal to one. If the re-normalized eigenvector is denoted by  $\bar{x}_r$  and it is assumed that its largest component is the  $m$ -th one, then Eq. (31) is replaced by

$$\bar{x}_{rm} = 1 \quad (36)$$

and Eq. (33) is replaced by

$$\frac{d\bar{x}_{rm}}{dp_j} = 0 \quad (37)$$

For  $\bar{x}_r$ , Eq. (32) reduces to

$$\left( K - \omega_r^2 M \right) \left( \frac{d\bar{x}_r}{dp_j} \right) = \left( \frac{d\omega_r^2}{dp_j} \right) M \bar{x}_r - \left( \frac{dK}{dp_j} - \omega_r^2 \frac{dM}{dp_j} \right) \bar{x}_r \quad (38)$$

Equation (37) is now used to reduce the order of Eqs. (38) by deleting the m-th row and m-th column. When the eigenvalue  $\omega_r^2$  is distinct, the reduced system is not singular and can be solved by a standard technique for the derivative vector  $d\bar{x}_r/dp_j$ . The required vectors  $x_r$  and  $dx_r/dp_j$  are then obtained from  $\bar{x}_r$  and  $d\bar{x}_r/dp_j$  by the following easily verified relations

$$x_r = \bar{x}_r \left( \bar{x}_r^T M \bar{x}_r \right)^{1/2} \quad (39)$$

and

$$\frac{dx_r}{dp_j} = \frac{\left( \frac{d\bar{x}_r}{dp_j} \right) \bar{x}_r \left[ \bar{x}_r^T \frac{dM}{dp_j} \bar{x}_r + 2 \bar{x}_r^T M \frac{d\bar{x}_r}{dp_j} \right]}{\left( \bar{x}_r^T M \bar{x}_r \right)^{1/2} + 2 \left( \bar{x}_r^T M \bar{x}_r \right)^{3/2}} \quad (40)$$

In finite element computer codes that exploit the sparsity structure of the K and M matrices, it may be inconvenient to obtain the re-normalized vector,  $\bar{x}_r$ , by setting the largest component to unity. Such a scheme necessitates the recalculation of the sparsity structure. Instead, it is more convenient to obtain the re-normalized eigenvector,  $\bar{x}_r$ , by setting

$$\bar{x}_{rn} = 1 \quad (41)$$

where n is the order of the matrices K and M.

As mentioned previously, the derivative of the eigenvector  $x_r$  with respect to the design variable  $p_j$  can also be calculated by the forward difference scheme

$$\frac{dx_r}{dp_j} = \frac{(x_r)_{p_j+h} - (x_r)_{p_j}}{h} \quad (42)$$



where  $(x_r)_{p_{j+h}}$  is the eigenvector calculated at  $p_{j+h}$ . In order to assess the accuracy of the forward difference scheme relative to Nelson's analytical method, an error measure is defined as

$$\epsilon(h) = \sum_j \sum_r \sum_i \left( \left( \frac{\partial x_{ri}}{\partial p_j} \right)_A - \left( \frac{\partial x_{ri}}{\partial p_j} \right)_F \right)^2 \quad (43)$$

where  $\left( \frac{\partial x_{ri}}{\partial p_j} \right)_A$  and  $\left( \frac{\partial x_{ri}}{\partial p_j} \right)_F$  are the eigenvector derivatives by the analytical and the forward difference scheme respectively. The error  $\epsilon$  is summed over all the components of the eigenvector, over the mode shapes controlled and over all the design variables.

### Application to a Stiffened Composite Plate

A laminated composite square plate reinforced by two stiffeners placed symmetrically with respect to the laminate midplane along the two centerlines of the plate is considered. As in reference [6] the plate is discretized using a mesh of 8 noded isoparametric, shear deformable plate bending elements. Assuming the plate is simply-supported along all its four edges, the resulting finite-element model has a respectable (from control engineer's point of view) 127 degrees of freedom and thirteen design variables consisting of five discrete fiber orientations and eight continuous stiffener cross-sectional areas.

Table 1 provides an assessment of the error  $\epsilon$  as a function of the step size  $h$  for the finite difference derivative calculations. As expected, the error decreases with a decrease in  $h$  and then begins to increase as a result of machine roundoff.

A comparison was made of the computational cost for the calculation of the eigenvector derivatives using Nelson's method and the finite difference scheme. Using Nelson's method to compute the gradient of the three eigenvectors with respect to the thirteen design variables the required CPU time was 17.2 seconds. To compute the eigenvector gradients using forward differences several eigenvalue problems need to be solved. Using subspace iteration in conjunction with the Jacobi method [7] for the solution of the eigenvalue problem, the total time for the required gradient calculations was 39.5 seconds. Note that the design vector has thirteen variables, and it was necessary to solve the perturbed eigenvalue problem thirteen times. Since the solution of the unperturbed eigenvalue problem provides an excellent guess for the eigenvalue of the perturbed system, an inverse iteration scheme [7] in conjunction with shifting of the stiffness matrix  $K$  can be used to accelerate the solution process. Using such a strategy, the CPU time required for the calculation of the eigenvector gradients using forward differences was down to 27.6 seconds.

Thus, in relation to the analytical method the computational cost of the finite difference calculation of the eigenvector gradients is not at all prohibitive. On the other hand, in spite of this modestly higher computational cost, the simplicity of the calculation of the eigenvector gradients using forward difference scheme is overwhelming. However some caution must be exercised when using inverse iteration in conjunction with shifting of the stiffness matrix. It should be noted that the normalization scheme  $x_r^T M x_r = 1$  fixes only the magnitude of the eigenvector and if  $x_r$  is an eigenvector, then  $-x_r$  is also an eigenvector of the system. Hence, when eigenvectors of the perturbed system are computed, care must be taken to choose the eigenvector  $(x_r)_{p_j+h}$  such that

$$(x_r)_{p_j+h}^T (x_r)_{p_j} > 0$$

This can be done very easily in practice by simply calculating the above dot product and changing the sign of the vector  $x_r$  if the dot product is negative.

**Table 1. Error  $\epsilon$  as a Function of the Step Size  $h$**

$h$	$\epsilon$
$0.156 \times 10^{-1}$	$0.79 \times 10^{-6}$
$0.781 \times 10^{-2}$	$0.21 \times 10^{-6}$
$0.390 \times 10^{-2}$	$0.54 \times 10^{-7}$
$0.195 \times 10^{-2}$	$0.13 \times 10^{-7}$
$0.976 \times 10^{-3}$	$0.34 \times 10^{-8}$
$0.488 \times 10^{-3}$	$0.98 \times 10^{-9}$
$0.244 \times 10^{-3}$	$0.80 \times 10^{-9}$
$0.122 \times 10^{-3}$	$0.25 \times 10^{-8}$
$0.610 \times 10^{-4}$	$0.101 \times 10^{-7}$
$0.305 \times 10^{-4}$	$0.404 \times 10^{-7}$
$0.152 \times 10^{-4}$	$0.162 \times 10^{-6}$

Table 2 provides a comparison of the computational effectiveness of the two approaches for the control of the stiffened laminated composite plate problem for cases involving different number of design variables and different number of frequencies being controlled.

**Table 2. Finite Difference Versus Nelson' Approach - Normalized CPU Time**

Type of Design	Finite Difference Approach	Nelson's Approach
13 design variable, 3 frequencies	1.598	1.0
13 design variables, 8 frequencies	1.939	1.0
5 design variables, 8 frequencies	1.700	1.0

It is clear from Table 2 that in all the cases considered the finite difference approach requires more CPU time as compared to the Nelson's Analytic approach. The percentage increase in CPU time increases with the number of frequencies considered. In the finite difference approach an eigenvalue problem needs to be solved for each design variable considered. If shifting the K matrix in conjunction with inverse iteration is used to calculate the eigenvalues/eigenvectors, the finite difference approach is quite competitive with the Nelson's analytic approach. Even though the approach may require about twice the time of Nelson's method, the coding effort is far less in the case of the finite difference approach. Secondly, calculation of  $\frac{\partial K}{\partial p}$ ,  $\frac{\partial M}{\partial p}$  (derivatives of stiffness and mass matrices respectively) required in the case of Nelson's approach can be quite difficult in some cases. In the case where the design variables are element frame areas, calculation of  $\frac{\partial K}{\partial p}$ ,  $\frac{\partial M}{\partial p}$  is fairly straightforward. However, if p, corresponds to the number of plies with a given orientation then the calculation of  $\frac{\partial K}{\partial p}$ ,  $\frac{\partial M}{\partial p}$  is fairly involved.

In conclusion, it needs to be emphasized that the finite difference scheme for the calculation of the eigenvalue and eigenvector derivatives does not appear to be costly enough to warrant the use of the analytical method. With the former scheme one does not have to "tinker" with the "black box" that generates the eigenvalues and eigenvectors for a given design variable vector. The analytical method on the other hand needs an intimate knowledge of this "black box".

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